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COMMENT

The equiscale transformation in the same universality class

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Abstract. A new concept is proposed, termed the equiscale transformation in the same universality class. Application to the square lattice Ising model with first- and second-neighbour interactions yields critical lines which are in very satisfactory agreement with series results.

Renormalisation group theory (Wilson and Kogut 1974, Wilson 1975, Niemeijer and van Leeuwen 1976) has been extensively used in the study of phase transitions and critical phenomena in recent years. The essence of the application of the renormalisation group to systems with many degrees of freedom is a stepwise evaluation of the free energy. A renormalisation transformation is a scale transformation that leaves the partition function invariant. In a recent letter (Tang and Hu 1986a), we proposed a new idea which is now termed the equiscale transformation in the same universality class. With the help of this transformation, the exact decimation transformation, which makes the interaction parameter space large, has produced very good results for the Ising model (Tang and Hu 1986a), the Potts model (Ye *et al* 1987) and bond-diluted Ising magnets (Tang and Hu 1987). Since the exact renormalisation transformations preserve all the symmetry properties, a four-branch critical line is found for a anisotropic Ising model. These branches corrrespond to the ferromagnetic (F) phase ($K_x>0$, $K_y>0$), antiferromagnetic (AF) phase ($K_x<0$, $K_y<0$) and the two super-antiferromagnetic phases ($K_x>0$, $K_y<0$ and $K_x<0$, $K_y>0$) respectively (Tang and Hu 1986b). This transformation has also been used to improve the Migdal–Kadanoff renormalisation approach by finding the relation between the coupling constants ($\tilde{K}$ and K) of the bond-diluted lattice and the original one, instead of the MK approximation $\tilde{K}/K=2$, and calculations of the critical couplings and exponents for the two- and three-dimensional Ising models show much improvement compared with the results of MK calculations (Tang *et al* 1987). In this comment, we show another use of this transformation approach and determine the critical surface for some unsolvable systems according to the results of solvable systems which belong to the same universality as these unsolvable ones.

In order to illustrate our approach, we consider a square-lattice Ising model with first- and second-neighbour interactions described by the effective Hamiltonian

$$H = K_1 \sum_{\mathrm{nn}} S_i S_j + K_2 \sum_{\mathrm{nnn}} S_i S_j \tag{1}$$

where the summations are over nearest-neighbour pairs and next-nearest-neighbour pairs respectively, and K_1 and K_2 are coupling constants which may be either positive or negative. The inclusion of second-neighbour interactions makes the model unsolvable by existing techniques. Although no exact results are known, various approximate methods, including closed-form approximations (Fan and Wu 1969, Burkhardt 1978, Gibberd 1969), renormalisation groups (Nauenberg and Nienhuis 1974, van Leeuwen 1975, Nightingale 1977), series expansions (Oitmaa 1981) and Monte Carlo approaches (Landau 1980, Binder and Landau 1980) have been used to study the model and our overall knowledge of its properties is good. The model may be in an F state, an AF state and an SAF state, depending on the values of the interaction parameters K_1 and K_2. Therefore, the critical lines have three branches (see figure 1). The upper two branches, which represent transitions from the high-temperature paramagnetic phase (P) to either the F or AF phases, intersect the $K_2=0$ axis at $K_1=\pm K_0$ (where $K_0=\frac{1}{2}\ln(1+\sqrt{2})$ is the Onsager value), and come together in a cusp at the point $K_1=0$, $K_2=K_0$. It is believed that along these branches the system exhibits conventional Ising critical behaviour. The low branch which represents transitions from the P phase to the SAF phase intersects the $K_1=0$ axis at $K_2=-K_0$. there is evidence that along this line the system has non-universal exponents.

The equiscale transformation, based on the equivalence of critical behaviour of Hamiltonians which belong to the same universality class but have different form, can be performed by a mean-field-like approximation. Let us consider a finite cluster with L^2 spins. For this cluster, the order parameter can be computed in the presence of symmetry-breaking boundary conditions, which, in a mean-field sense, simulate the effect of surrounding spins in infinite extensions of the finite cluster. The interactions

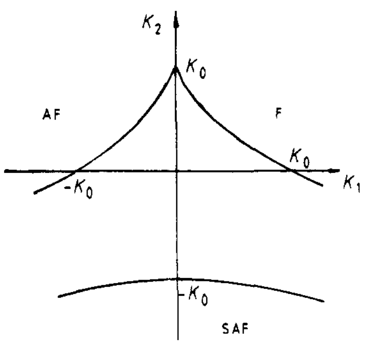

Figure 1. Qualitative shape of the phase diagram for the square-lattice Ising model with first- and second-neighbour interactions.

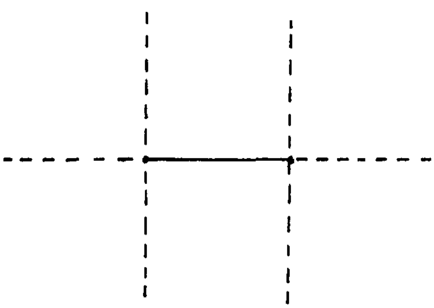

Figure 2. Schematic representation of the cluster ($L=\sqrt{2}$). Broken bonds represent interactions with symmetry-breaking boundary fields.

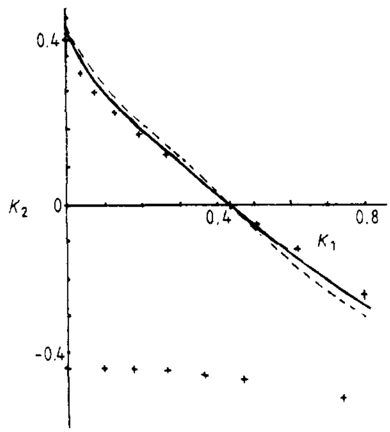

Figure 3. The phase diagram in the K_1 and K_2 plane for the square-lattice Ising model with first- and second-neighbour interactions. The results for $L=\sqrt{2}$, $L=2$ and series (Oitmaa 1981) are shown by a broken curve, full curve and crosses, respectively.

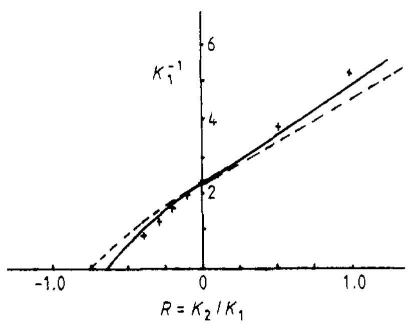

Figure 4. Variation of the critical temperature (K_1^{-1}) for the F transition, as a function of $R=K_2/K_1$. The results for $L=\sqrt{2}$ (broken curve), $L=2$ (full curve) and the series expansion (Oitmaa 1981) (crosses) are shown.

between the spins inside the cluster are treated exactly, whereas the interactions between internal spins and external ones are modified by fixing the external spins to a value m. The equations of critical surface are obtained by imposing

$$M_1 = qM_2 \tag{2}$$

$$m_1 = qm_2 \tag{3}$$

to leading order in m for m approaching zero, where M_1 and M_2 are the order parameters for two systems respectively. In the present case, M_1 and M_2 may be the average magnetisation corresponding to the F state, or the staggered magnetisation corresponding to the AF state. M_1 is computed for the Hamiltonian (1) with general K_1 and K_2 and M_2 for the Hamiltonian (1) with special interaction parameters ($K_2=0$, $K_1=\pm K$) for which the exact critical points are known. For the cluster shown in figure 2 ($L=\sqrt{2}$), the equations of the critical line in the (K_1, K_2) plane are

$$\text{F:}\quad K_2 = \tfrac{3}{4}\{M_0[1+\exp(-2K_1)] - K_1\} \tag{4}$$

$$\text{AF:}\quad K_2 = \tfrac{3}{4}\{M_0[1+\exp(2K_1)] + K_1\} \tag{5}$$

$$M_0 = \tfrac{1}{2}(1+\sqrt{2})[\ln(1+\sqrt{2})]/(2+\sqrt{2}) \tag{6}$$

which is shown in figure 3 with the results of $L=2$ and the high-temperature series expansion (Oitmaa 1981). Since in the SAF phase the system has non-universal exponents, the low line in figure 1 cannot be obtained by our method. Our calculations are in good agreement with series results even for small clusters and tend to be exact when $L \to \infty$.

Another interesting problem is the behaviour of this model when $K_2 = -\frac{1}{2}K_1$ since in this case the ground state is highly degenerate. The series expansion of Oitmaa is irregular for $R(=K_2/K_1)$ in the range $-0.5 \leqslant R < -0.4$ and it is unable to determine whether T_c goes to zero at $R = -\frac{1}{2}$ or whether it approaches a small finite limit. In our calculations there is an R_0 at which T_c goes to zero and $R(L) = -L/(2L-1)$ for general L. As is well known, the mean-field-like approximation tends to be exact when $L \to \infty$. Hence, we obtain $R_0 = -\frac{1}{2}$, which may be an exact result and is consistent with the suggestion of Oitmaa. The variation of ferromagnetic critical temperature with R is plotted in figure 4. For comparison, the series expansion results of Oitmaa (1981) are also shown.

In conclusion, the new concept of equiscale transformation in the same universality class is very useful for phase transitions and critical phenomena. The new use of this idea is shown and the application to the square-lattice Ising model with first- and second-neighbour interactions yields the critical lines which are in very satisfactory agreement with series results.

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